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Equation of state for exclusion statistics in a harmonic well

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Abstract. We consider the equations of state for systems of particles with exclusion statistics in a harmonic well. Paradigmatic examples are non-interacting particles obeying ideal fractional exclusion statistics placed in (i) a harmonic well on a line, and (ii) a harmonic well in the lowest Landau level (LLL) of an exterior magnetic field. We show their identity with (i) the Calogero model and (ii) anyons in the LLL of an exterior magnetic field and in a harmonic well.

Haldane's definition of fractional statistics based on a generalized exclusion principle [1] when applied *locally* in phase space results in single state statistical distributions for *ideal* fractional exclusion statistics (IFES) [2, 3]. Thermodynamic quantities corresponding to IFES first appeared in the literature for anyons in the lowest Landau level (LLL) of an exterior magnetic field [4], i.e. for a gas of particles with a degenerate one body spectrum. On the other hand, in one dimension, IFES can be modelled with inverse square interactions [5]. The relevant systems are the Sutherland model (particles on a circle) [6] and the Calogero model (particles on a line in a harmonic potential) [7].

To derive the equation of state (or the virial expansion) for interacting systems modelling IFES [8, 9], one can start from a system placed in a harmonic potential and then use a thermodynamic limit prescription to obtain the equation of state for the original system in the infinite box. The harmonic potential is then referred to as a 'long distance regulator'. This procedure, originally proposed for anyons [10], has then been put on general ground [11, 8]. In particular, it is precisely the way that the equation of state for anyons in the LLL was originally derived [4]. (For several species of particles see [12].) The same procedure can be applied to 1D particles with inverse square interactions: virial coefficients for this model have been derived [8] starting from the energy levels of the Calogero model and using an adequate 1D thermodynamic limit prescription. The latter result was also verified using the equivalence of the Sutherland model to a system of free particles obeying IFES [13].

In this paper, we address the question of the equation of state for exclusion statistics in a harmonic potential, without considering the transition to a system in the infinite box limit. We refer to this case as a 'physical' harmonic potential. What we have in mind are

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mesoscopic quantum dots systems where finite volume effects are supposed to be described by a harmonic well. Since there is no well-defined volume in this system, the equation of state (or the potential Ω) depends on the particle number N, the harmonic potential frequency ω, and on other parameters as well. A similar form of the equation of state has already been discussed in the Thomas–Fermi approximation for 1D IFES particles in an external potential [14].

We start by assuming a certain temperature scaling for the one-particle partition function of non-interacting IFES particles. In this context, the equation of state is valid both for particles in an infinite box and in a 'physical' harmonic well. As examples, we discuss non-interacting particles in (i) a 1D harmonic well and (ii) in a 2D harmonic well in the LLL of an external magnetic field. Both these cases correspond to constant density of levels in energy. We compare the relevant equations of state with those for (i) the Calogero model and (ii) anyons in a harmonic well in the LLL of an external magnetic field. We discuss more generally a specific form of the equation of state for IFES with a constant density of states for systems both in a box and in a harmonic well. Introducing the effective volume occupied by the gas in a harmonic well at a given temperature, we obtain finally an equation of state in a 'physical' harmonic well quite similar to those in a box.

Ideal fractional exclusion statistics can be defined by the single-state grand partition function

$$[\xi(x_i)]^{g-1}[\xi(x_i)-1] = x_i \tag{1}$$

where ε_i is the energy of the state i and x_i is the Gibbs factor $x_i \equiv \mathrm{e}^{\beta(\mu-\varepsilon_i)}$, $\beta = 1/k_{\mathrm{B}}T$.

The statistics parameter $g = 0$ corresponds to Bose and $g = 1$ to Fermi statistics. The distribution function $n(x_i) = x_i(\partial/\partial x_i)\ln\xi(x_i)$ is connected with $\xi(x_i)$ by the bilinear relation

$$\frac{1}{n(x_i)} = \frac{1}{\xi(x_i)-1} + g. \tag{2}$$

Expanding in powers of x_i yields

$$\ln\xi(x_i) = \sum_{k=1}^{\infty}\frac{Q_k}{k}x_i^k \qquad Q_k = \prod_{l=1}^{k-1}\left(1-g\frac{k}{l}\right) = \frac{\Gamma(k-gk)}{\Gamma(k)\Gamma(1-k)}. \tag{3}$$

By summing $\ln\xi(x_i)$ and $n(x_i)$ over i, one obtains the expansions

$$-\beta\Omega = \sum_{k=1}^{\infty} b_k z^k \qquad N = \sum_{k=1}^{\infty} k b_k z^k \tag{4}$$

($z = \mathrm{e}^{\beta\mu}$ is the fugacity), with the cluster coefficients

$$b_k = \frac{Q_k}{k} Z_1(k\beta) \tag{5}$$

$Z_1(\beta) = \sum_i \mathrm{e}^{-\beta\varepsilon_i}$ is the one-particle partition function.

Without specifying in detail the system at this stage, we assume that the partition function $Z_1(\beta)$ scales with the inverse temperature β as

$$Z_1(k\beta) \simeq \frac{\mathrm{e}^{-k\beta\varepsilon_0}}{k^{1+\delta}} Z_1'(\beta). \tag{6}$$

This scaling is relevant for systems with a gap ε_0 in the single particle energy spectrum, with one-particle partition function $Z_1(\beta) = \mathrm{e}^{-\beta\varepsilon_0}Z_1'(\beta)$. This factorization will indeed materialize in the thermodynamic limit for various physical systems as we will see below.

Taking into account (6), the cluster coefficients (5) become

$$b_k = \frac{Q_k}{k^{2+\delta}} \mathrm{e}^{-k\beta\varepsilon_0} Z_1'(\beta). \tag{7}$$

One deduces from (4) the 'virial expansion'

$$-\beta\Omega = \sum_{k=1}^{\infty} A_k \frac{N^k}{[Z_1'(\beta)]^{k-1}} \tag{8}$$

with the dimensionless 'virial coefficients'

$$A_1 = 1 \qquad A_2 = \frac{1}{2^{2+\delta}}(2g-1)$$
$$A_3 = [(4^{-1-\delta} - 2\times 3^{-2-\delta}) + g(g-1)(4^{-\delta} - 3^{-\delta})], \ldots. \tag{9}$$

We now specialize on the particular case $\delta = 0$, which can be completely analysed. This case is of particular interest since it implies a $1/\beta$ scaling for $Z_1(\beta)$, which means a constant density of states in energy. One can already see from (9) that $\delta = 0$ is special since it implies that A_3 does not depend on the statistical parameter g.

Comparing the second expansion in (4) with the cluster coefficients (7), with (3), one can write

$$N = Z_1'(\beta)\ln\xi(z') \tag{10}$$

where $z' = z\mathrm{e}^{-\beta\varepsilon_0}$. Regarding equation (10) as determining z' as a function of N, we obtain from (4)

$$-\frac{\partial\beta\Omega}{\partial N} = \frac{1}{z}\frac{\partial z'(N)}{\partial N}N. \tag{11}$$

To calculate $\partial z'(N)/\partial N$ (or $\partial N(z')/\partial z'$), we use (10) and (1), (2). Noting that $(1/N(z'))\partial N(z')/\partial z' = n(z')/z'$, we find an equation of state

$$-\frac{\partial\beta\Omega}{\partial N} = \frac{N}{Z_1'(\beta)}\left(\frac{1}{\mathrm{e}^{N/Z_1'(\beta)} - 1} + g\right) \tag{12}$$

and upon integration, finally,

$$-\beta\Omega = \int_0^N \frac{N'}{Z_1'(\beta)}\frac{\mathrm{d}N'}{(\mathrm{e}^{N'/Z_1'(\beta)} - 1)} + \frac{1}{2}g\frac{N^2}{Z_1'(\beta)}. \tag{13}$$

Expanding this, we obtain the virial expansion

$$-\beta\Omega = N\left\{1 + \frac{1}{4}(2g-1)\frac{N}{Z_1'(\beta)} + \sum_{k=2}^{\infty}\frac{\mathcal{B}_k}{(k+1)!}\left(\frac{N}{Z_1'(\beta)}\right)^k\right\} \tag{14}$$

where $\mathcal{B}_k$ are the Bernoulli numbers ($\mathcal{B}_2 = \frac{1}{6}$, $\mathcal{B}_4 = -\frac{1}{30}$), vanishing for k odd. It appears that the statistical parameter g only enters the equation of state via the second virial coefficient [15].

Let us now examine how the equations of state (8) and (13) can be relevant both to systems in a box and in a 'physical' harmonic well.

We first consider a gas of free (spinless) particles, with generic dispersion law $\varepsilon(p) = \varepsilon_0 + ap^\sigma$ occupying a box of volume V in d dimensions. In the thermodynamic limit the one-particle partition function is

$$Z_1(k\beta) = \mathrm{e}^{-k\beta\varepsilon_0}\frac{\Gamma(1+d/\sigma)V}{(2\sqrt{\pi})^d\Gamma(1+\frac{1}{2}d)(ak\beta)^{-d/\sigma}} \tag{15}$$

satisfying (6), with $\delta = d/\sigma - 1$. In this case the virial expansion (8) becomes the usual virial expansion for a system in a box with pressure P given by

$$\beta P = \sum_{k=1}^{\infty} a_k \rho^k \qquad \rho = N/V \tag{16}$$

with the (dimensional) virial coefficients

$$a_k = A_k \left(\frac{V}{Z_1'(\beta)} \right)^{k-1} \tag{17}$$

where A_k are given in (9). Note that the virial coefficients (17) coincide with those obtained in [13] for $\varepsilon_0 = 0$ showing that the presence of a gap in the particle dispersion does not affect the equation of state.

The constant density of states case, $\delta = d/\sigma - 1 = 0$, describes for example chiral particles on a line with linear dispersion with a gap,

$$\varepsilon = \varepsilon_0 + vp \qquad p \geqslant 0. \tag{18}$$

The equation of state is (13) (or (14)) with

$$Z_1'(\beta) = \frac{V}{2\pi v \beta}. \tag{19}$$

Let us now turn to systems in a 'physical' harmonic well. We restrict ourselves to cases with a constant density of states, i.e. we consider non-interacting (non-relativistic) particles that occupy single particle levels:

(i) in a 1D harmonic well $V(x) = \frac{1}{2} m\omega^2 x^2$;

(ii) in the LLL of an external magnetic field B in a 2D harmonic well.

In a quantum dot language, where ω is small (for example with respect to the cyclotron frequency), but non-vanishing, the harmonic well is supposed to encode the finite size effects of the sample.

The one-particle energy levels are

$$\varepsilon_\ell = \varepsilon_0 + \ell \varpi \qquad \ell \geqslant 0 \tag{20}$$

where for the 1D harmonic model

$$\varepsilon_0 = \tfrac{1}{2}\omega \qquad \varpi = \omega \tag{21}$$

whereas for the 2D LLL harmonic model, one has

$$\varepsilon_0 = \sqrt{\omega_c^2 + \omega^2} \qquad \varpi = \sqrt{\omega_c^2 + \omega^2} - \omega_c \tag{22}$$

or, to leading order in ω^2/ω_c^2,

$$\varepsilon_0 = \omega_c \qquad \varpi = \frac{\omega^2}{2\omega_c}. \tag{23}$$

It follows from (20) that the one-particle partition function is

$$Z_1(k\beta) = \frac{e^{-k\beta\varepsilon_0}}{1 - e^{-k\beta\varpi}} = \frac{e^{-k\beta\varepsilon_0}}{k\beta\varpi}\left(1 + \tfrac{1}{2}(k\beta\varpi) + \mathcal{O}[(k\beta\varpi)^2]\right). \tag{24}$$

Thus, to leading order in $k\beta\varpi$, the partition function (24) scales as in (6), with $\delta = 0$ and

$$Z_1'(\beta) = 1/\beta\varpi. \tag{25}$$

This leads to the equation of state (13), (14). If one considers the correction terms $\mathcal{O}(k\beta\varpi)$ in (24), they might lead to corrections to the virial coefficients of very high order, N and above. However, if the virial expansion converges, these corrections are negligible inside the radius of convergence.

It is interesting to note that for 1D IFES particles in a harmonic well, the equation of state (13), (14) with (25) and (21) coincides with that obtained in the Thomas–Fermi approximation [14]. We stress that the present derivation only uses the condition $\beta\omega \ll 1$ when the discreteness of energy levels becomes inessential.

Systems modelling IFES are known to be 1D particles with inverse square interactions and 2D anyons in the LLL of an external magnetic field. A harmonic well may be used as a long distance regulator for these systems. We now investigate what happens if the harmonic well becomes 'physical' (the first system is then the Calogero model). The N-particle energy levels for both the systems are given by [8]

$$\sum_{\ell} n_\ell \varepsilon_\ell + \tfrac{1}{2} N(N-1)\varpi g \tag{26}$$

where n_ℓ are non-negative integers with the constraint $\sum_\ell n_\ell = N$, and ε_ℓ is given by (20). For anyons, $g = \alpha \in [0, 1]$ is the anyonic statistical parameter; for the Calogero model, $g = \alpha \geqslant 0$ defines the singular two-body potential $\alpha(\alpha - 1)/(x_i - x_j)^2$ and specifies the coinciding point (short distance) behaviour of the N-body wavefunction, $\Psi \propto |x_i - x_j|^\alpha$ as $|x_i - x_j| \to 0$.

The N-particle partition function corresponding to (26) is

$$Z_N = \mathrm{e}^{-\frac{1}{2}\beta N(N-1)\varpi g} \prod_{n=1}^{N} \frac{\mathrm{e}^{-\beta\varepsilon_0}}{1 - \mathrm{e}^{-n\beta\varpi}}. \tag{27}$$

Remarkably, the cluster coefficients obtained form (27) are [8]

$$b_k = e^{-k\beta\varepsilon_0} \frac{Q_k}{k^2 \beta\varpi} (1 + \mathcal{O}(\beta\varpi)) \tag{28}$$

having, to leading order in $\beta\varpi$, the form (7) with $\delta = 0$ and $Z_1'(\beta)$ given by (25). As above, the terms of order $\mathcal{O}(\beta\varpi)$ in (28) are negligible inside the radius of convergence of the virial expansion. It follows that both equations of state for the Calogero and LLL anyon model in a 2D harmonic well are given by (13), (14) with $Z_1'(\beta)$ determined by (25), i.e. they are identical to the equations of states for non-interacting IFES particles in a 1D harmonic well and in a 2D LLL harmonic well, respectively.

This happens to be a general feature of equations of states in a 'physical' harmonic well for various physical systems modelling IFES with a constant density of levels: only the second virial coefficient depends on the statistics parameter, in agreement with the general statements for free IFES particles in a box [13].

As one more example corresponding to a constant density of levels, consider the model of a chiral field on a circle proposed in [16]. This model was constructed by mapping the second quantized LLL anyon model onto a circle. The energy levels of this model are given by (26) with the identification

$$\frac{\omega^2}{2\omega_c} = \frac{2\pi}{V} v \tag{29}$$

where V is the length of the circle. If the harmonic potential is assumed to materialize in the anyon droplet of radius $V/2\pi$ by the action of an electric field, the velocity v can then be interpreted as the drift velocity E/B on the edge (the velocity of the edge excitations), where the electric field on the edge is $E = (m/e)\omega^2 R$. The thermodynamic limit is understood as $\omega \to 0$, $V \to \infty$ whereas v is kept fixed. The spectrum (26), together with the identification (29), yield the cluster coefficients (7) with $\delta = 0$, $\varepsilon_0 = \omega_c$, and $Z_1'(\beta)$ given by (19). This leads to the equation of state (13), coinciding with that of free chiral 1D IFES particles with dispersion (18). This conclusion is consistent with those of [17], namely, the model of a chiral field on a circle [16] admits an interpretation in terms of IFES.

We finally note that (13) (or (14)) for a 'physical' harmonic well, with $Z_1'(\beta)$ given by (25), can be viewed as an equation of state relating the average pressure to the average particle density in the harmonic well. The local pressure $P(\boldsymbol{x})$ in a slowly varying d-dimensional harmonic potential is defined as

$$\beta\Omega = \int \mathrm{d}^d\boldsymbol{x}\,\beta P(\boldsymbol{x}). \tag{30}$$

The average pressure is then

$$\beta\langle P\rangle \equiv \left(\frac{\beta\omega}{\lambda_T}\right)^d \int \mathrm{d}^d\boldsymbol{x}\,\beta P(\boldsymbol{x}) \tag{31}$$

where $\lambda_T = \sqrt{2\pi\beta/m}$ is the thermal de Broglie wavelength. The local particle density $\rho(\mathbf{x})$ being normalized as $\int \mathrm{d}^D\boldsymbol{x}\,\rho(\boldsymbol{x}) = N$, one can define the average density as

$$\langle\rho\rangle \equiv \left(\frac{\beta\omega}{\lambda_T}\right)^d \int \mathrm{d}^d\boldsymbol{x}\,\rho(\boldsymbol{x}). \tag{32}$$

Introducing the effective volume occupied by the gas at temperature T by

$$V_{\mathrm{eff}} = \left(\frac{\lambda_T}{\beta\omega}\right)^d \tag{33}$$

we have $V_{\mathrm{eff}} \sim R_{\mathrm{eff}}^d$, where R_{eff} is determined by

$$\tfrac{1}{2}m\omega^2 R_{\mathrm{eff}}^2 \simeq T. \tag{34}$$

It follows that (14) can then be rewritten as a virial expansion in a box of volume V_{eff} (cf (16), (17)):

$$\beta\langle P\rangle = \langle\rho\rangle\left(1 + \tfrac{1}{4}(2g-1)\beta\varpi V_{\mathrm{eff}}\langle\rho\rangle + \sum_{k=2}^{\infty}\frac{\mathcal{B}_k}{(k+1)!}(\beta\varpi V_{\mathrm{eff}})^k\langle\rho\rangle^k\right). \tag{35}$$

For non-interacting particles in a 1D harmonic well and for the Calogero model, $\beta\varpi V_{\mathrm{eff}} = \lambda_T$, whereas for non-interacting LLL particles in a 2D harmonic well and for LLL anyons in a 2D harmonic well, $\beta\varpi V_{\mathrm{eff}} = 2\pi/eB$. For the Calogero model, the equation of state in the form (35) was conjectured in [8]. Note that for the LLL anyon model, the condition $1 \ll 2\beta\omega_c$ guarantees that the particles, when confined in the quantum dot of size R_{eff} given by (34), do not reach the second Landau level.

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References

[1] Haldane F D M 1991 *Phys. Rev. Lett.* **67** 937
[2] Isakov S B 1994 *Mod. Phys. Lett.* B **8** 319
[3] Wu Y S 1994 *Phys. Rev. Lett.* **73** 922
[4] Dasnières de Veigy A and Ouvry S 1994 *Phys. Rev. Lett.* **72** 600

[5] Isakov S B *Int. J. Mod. Phys.* A **9** 2563
Bernard D and Wu Y S 1994 *Preprint* cond-mat/9404025
Ha Z N C 1994 *Phys. Rev. Lett.* **73** 1574; 1995 *Nucl. Phys.* B **435** 604
Murthy M V N and Shankar R 1994 *Phys. Rev. Lett.* **73** 3331

[6] Sutherland B 1971 *J. Math. Phys.* **12** 246, 251; 1971 *Phys. Rev.* A **4** 2019; 1972 *Phys. Rev.* A **5** 1372

[7] Calogero F 1969 *J. Math. Phys.* **10** 2191, 2197; 1971 *J. Math. Phys.* **12** 419

[8] Dasnières de Veigy A and Ouvry S 1995 *Phys. Rev. Lett.* **75** 352; 1995 *Mod. Phys. Lett.* B **9** 271

[9] Murthy M V N and Shankar R 1995 *Phys. Rev. Lett.* **75** 353

[10] Comtet A, Georgelin Y and Ouvry S 1989 *J. Phys. A: Math. Gen.* **22** 3917; see also the second reference in [6]

[11] Olaussen K 1992 *Theor. Phys. Seminar in Trondheim* No 13 (unpublished), cond-mat/9207005

[12] Isakov S B, Mashkevich S and Ouvry S 1995 *Nucl. Phys.* B **448** 457

[13] Isakov S B, Arovas D P, Myrheim J and Polychronakos A P 1996 *Phys. Lett.* **212A** 299

[14] Sen D and Bhaduri R K 1995 *Phys. Rev. Lett.* **74** 3912

[15] For discussions of systems where only the second virial coefficient is affected by the statistics, see Dasnières de Veigy A 1996 *Nucl. Phys.* B **458** 533; see also [9]

[16] Hansson T H, Leinaas J M and Viefers S 1996 *Nucl. Phys.* B **470** 291

[17] Isakov S B and Viefers S 1996 *Int. J. Mod. Phys.* A to be published; 1996 *Preprint* hep-th/9605157